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Introduction

In [1] M.J. Lighthill proposed a new perturbation technique for the approximate solution of certain types of nonlinear differential equations near their singular points. This method has since found numerous useful applications in hydrodynamical problems, an excellent account of which can be found in a report by H.S. Tsien [2]. The procedures are justified, in the existing literature, by very convincing plausibility considerations but mathematically complete proofs seem to be lacking.

The purpose of this note is to give such a proof in the simple but typical case of the differential equation

$$(x + \alpha u)\frac{du}{dx} + q(x)u - r(x) = 0, \qquad (1.1)$$

where α is a small positive parameter. This is the example used also in [1] for an introduction to the technique. In fact, our proof is essentially a completion of the order-of-magnitude arguments employed in that article. We shall therefore be somewhat brief here in our explanation of the formal aspects of the procedure.

The problem is to represent the solution of (1.1) through a point

$$x = a, \qquad u = b \tag{1.2}$$

in a form that remains valid near $x = 0$, $\alpha = 0$. This is achieved by referring x and u to a suitably chosen parameter ξ, as will be explained in the next section.

2. Construction of the Solution.

We make the following assumptions concerning the differential equation (1.1) and the prescribed initial values (1.2):

(i) $\quad \alpha > 0 , \qquad a > 0 , \qquad b > 0 .$ (2.1)

(ii) The functions $r(x)$, $q(x)$ are holomorphic in $|x| \leq a$.

Without loss of generality we may set

$$a = 1 , \tag{2.2}$$

since this can be brought about by the transformation $x = x^* a$, $\alpha = \alpha^* a$.

We set, formally,

$$x = \xi + x_1 \alpha + x_2 \alpha^2 + \ldots \tag{2.3}$$

$$u = u_o + u_1 \alpha + u_2 \alpha^2 + \ldots , \tag{2.4}$$

where the u_r and x_r are functions of ξ to be determined and insert this into the equation

$$(x + \alpha u)u' + (q(x)u - r(x))x' = 0 , \qquad (2.5)$$

equivalent with (1.1) . If $r(x)$ and $q(x)$ are developed in powers of $x - \xi$ about the point ξ , setting

$$r(x) = r(\xi) + r'(\xi)(x - \xi) + \frac{1}{2} r''(\xi)(x - \xi)^2 + \dots$$

etc., the left member of (2.5) becomes, after collection of like powers of α, a series of the form

$$L_o + L_1\alpha + L_2\alpha^2 + \dots , \qquad (2.6)$$

whose coefficients are fairly complicated expressions in the x_j , u_j , x_j' , u_j' and ξ . (We denote differentiation with respect to ξ by dashes.) More precisely, we find (writing q,r for $q(\xi)$, $r(\xi)$),

$$L_o \equiv \xi u_o' + qu_o - r , \qquad (2.7)$$

$$L_s \equiv \xi u_s' + qu_s + (qu_o - r)x_s' + (u_o' + q'u_o - r')x_s - R_s, \quad s > 0 \qquad (2.8)$$

where

$$R_s = -\sum_{\substack{\mu+\nu=s\\ \mu,\nu>0}} x_\mu u_\nu' - \sum_{\mu+\nu=s-1} u_\mu u_\nu' + q \sum_{\substack{\mu+\nu=s\\ \mu,\nu>0}} x_\mu' u_\nu - q' \sum_{\substack{\mu+\nu+\rho=s\\ \mu,\nu>0}} x_\mu x_\nu' u_\rho \tag{2.9}$$

$$- h_s(x_1,\ldots,x_{s-1};\ x_1',\ldots;x'_{s-1};u_0,\ldots,u_{s-1};\xi), \quad s=1,2,\ldots$$

the function h_s being defined as follows: In the expression

$$h(x,u,x',\xi) = \Big\{-[q(x) - q - (x-\xi)q']u + [r(x) - r - (x-\xi)r']\Big\}x' \tag{2.10}$$

replace x and u by the series (2.3), (2.4), expand about $x = \xi$, and collect like powers of α. Then the coefficient of α^s is called h_s. The quantity h_s is therefore a polynomial in the x_j, x_j', u_j whose terms are of the form

$$A(\xi)x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_\mu^{\alpha_\mu} x_\nu'^{\beta} u_\rho^{\gamma} , \tag{2.11}$$

with

$$\alpha_1 + 2\alpha_2 + \ldots + \mu\alpha_\mu + \nu\beta + \rho\gamma = s , \tag{2.12}$$

$$\beta = 0,\ 1 , \qquad \gamma = 0,\ 1 .$$

Since the expansion of (2.10) in powers of $x - \xi$ contains no constant or linear terms, we know also that $\sum_{j=1}^{\mu} \alpha_j \geq 2$.

It follows that, if α_μ , β, or γ are positive, then

$$\mu \leq s - 1 , \quad \nu \leq s - 1 , \quad \rho \leq s - 1 .$$

In particular, h_1 is identically zero.

In order to satisfy the equation (1.1) the functions $x_s(\xi)$, $u_s(\xi)$ must now be successively determined in such a way that the L_s , as defined in (2.7) and (2.8), vanish. At the same time we must impose boundary conditions on these functions such that resulting series (2.3), (2.4) assume the values $x = 1$, $u = b$ for some $\xi = \xi_1$. Now, each equation $L_s = 0$, $s > 0$, can be replaced, in infinitely many ways, by two differential equations of the form

$$(qu_o - r)x_s' + (u_o' + q'u_o - r')x_s = R_s^{(1)} \tag{2.13}$$

$$\xi u_s' + qu_s = R_s^{(2)} \tag{2.14}$$

the only condition being that

$$R_s^{(1)} + R_s^{(2)} = R_s \quad . \tag{2.15}$$

This is a genuinely recursive system, since R_s depends only on the x_j , u_j with $j \leq s-1$.

The radius of convergence of the resulting series (2.3), (2.4) depends on ξ and on the manner in which each R_s is split into $R_s^{(1)}$ and $R_s^{(2)}$. In the next section it will be shown that an acceptable method, from this viewpoint, consists in taking $R_s^{(2)} = 0$ for all s. But there are other possibilities, which may offer computational advantages.

3. The Linear Recursion Equations.

In order to calculate u_o from the equation $L_o = 0$, (cf. formula (2.7)) we observe first that for $q_o = q(0) \neq 0$ the corresponding homogeneous equation is solved by

$$e^{-\int q(\xi)\xi^{-1}d\xi} = \xi^{-q_o} p(\xi) \quad ,$$

where $p(\xi)$ is holomorphic and different from zero in $|\xi| \leq 1$. Hence,

$$u_o(\xi) = c_o\xi^{-q_o}p(\xi) + \xi^{-q_o} p(\xi) \int_{\xi_1}^{\xi} p^{-1}(t)t^{q_o-1}r(t)dt. \quad (3.1)$$

The constant c_o depends on the choice of $u_o(\xi_1)$. For the sake of simplicity we set

$$\xi_1 = 1$$

$$u_o(1) = b \ , \quad u_s(1) = 0; \quad s > 0 \tag{3.2}$$

$$x_s(1) = 0 \ , \qquad s \geq 0 \tag{3.3}$$

and accordingly

$$c_o = b/p(1) \ .$$

Then the resulting functions $x(\xi)$, $u(\xi)$ will satisfy the boundary conditions $x(1) = 1$, $u(1) = b$.

It is seen from (3.1) that u_o has the form

$$u_o(\xi) = \xi^{-q_o} u_{o1}(\xi) + u_{o2}(\xi) \tag{3.4}$$

with $u_{o1}(\xi)$, $u_{o2}(\xi)$ holomorphic in $|\xi| \leq 1$.

In this section we shall assume that

$$q_o > 0 \ . \tag{3.5}$$

The case $q_o < 0$ will be discussed in section 6.

Next we calculate x_1 . Inspection of formula (2.9) shows that R_1 reduces to $u_o u_o'$. Following Lighthill <u>we require that $R_1^{(1)}$ contain the part of $u_o u_o'$ that involves negative powers of ξ higher than ξ^{-2q_o}</u>. Then we have

$$R_1^{(1)} = \xi^{-2q_o-1} r_{11}(\xi) \quad , \tag{3.6}$$

where $r_{11}(\xi)$ is a polynomial in ξ^{q_o} with coefficients that are holomorphic in ξ for $|\xi| \leq 1$. Inserting (3.6) and the expression for u_o and u_o' into (2.13) that equation is seen to be of the form

$$\xi e(\xi)x_1' + f(\xi)x_1 = \xi^{-q_o} r_{11}(\xi) \tag{3.7}$$

where $e(\xi)$ and $f(\xi)$ are linear functions of ξ^{q_o} with holomorphic coefficients. Furthermore,

$$\frac{f(\xi)}{e(\xi)} = \xi \frac{u_o' + q'u_o - r'}{qu_o - r} = \frac{\xi u_o'}{qu_o} (1 + O(\xi))$$

$$= (\frac{r}{qu_o} - 1)(1 + O(\xi)) = -1 + O(\xi) + O(\xi^{q_o})$$

At this point our procedure breaks down unless we impose a condition not mentioned in [1] or [2], viz.

$$q(\xi)\, u_o(\xi) - r(\xi) \neq 0 \,, \quad \text{in } 0 < \xi \leq 1 \,. \tag{3.8}$$

Then $e(\xi)$ does not vanish in this interval. Hence,

$$\exp \left\{ - \int \frac{f(\xi)}{\xi e(\xi)} \, d\xi \right\} = \xi \pi(\xi) \quad ,$$

where $\pi(\xi)$ is, in $0 \leq \xi \leq 1$, a holomorphic function of the two variables ξ and ξ^{q_o} without zeros.

The solution of (3.7) with $x_1(1) = 0$ is

$$x_1(\xi) = -\xi\pi(\xi)\int_1^\xi \pi^{-1}(t)t^{-q_o-2}\, r_{11}(t)dt = \xi^{-q_o}x_1^*(\xi). \qquad (3.9)$$

The function $x_1^*(\xi)$ is holomorphic in ξ and ξ^{q_o} .

For a fuller description of the stepwise calculation of the $x_s(\xi)$ and $u_s(\xi)$ we refer to [1] and [2] . At each step R_s must be split in such a way that $R_s^{(2)}$ is of the order $O(\xi^{-(s+1)q_o})$ as $\xi \to 0$. Any such choice of $R_s^{(1)}$ and $R_s^{(2)}$ will be called admissible. Then

$$R_s^{(1)} = O(\xi^{-(s+1)q_o-1}) , \qquad (3.10)$$

and

$$x_s(\xi) = \xi^{-sq_o}\, x_s^*(\xi) , \quad u_s(\xi) = \xi^{-(s+1)q_o}\, u_s^*(\xi) \qquad (3.11)$$

the functions $x_s^*(\xi)$, $u_s^*(\xi)$ being holomorphic in ξ and ξ^{q_o} .

These facts suggest the conjecture that the resulting series for $x(\xi)$ and $u(\xi)$ have radii of convergence of the order $O(\xi^{-q_o})$. However, in view of the degree of arbitrariness in the decomposition of R_s into $R_1^{(1)}$ and $R_s^{(2)}$ one can expect to prove such a convergence statement only on the

basis of some specific rule for this decomposition. If we wish to keep the choice of $R_s^{(1)}$ free, within the limitations imposed above, we must be content with a statement of asymptotic character.

The method to be used here is a combination of these two possibilities. We shall admit any admissible choice of $R^{(1)}$ for $s \leq m$, m arbitrary. For $s > m$ we shall set, however,

$$R_s^{(1)} = R_s \quad , \quad R_s^{(2)} = 0 \quad , \quad s > m \ . \qquad (3.12)$$

Hence, the series (2.4) terminates at the term in α^m , thanks to our choice of initial conditions.

As a preliminary to the convergence proof we derive two simple inequalities concerning the solutions of the differential equation problem

$$\xi e(\xi)v' + f(\xi)v = \phi(\xi) \quad , \quad v(1) = 0 \ , \qquad (3.13)$$

where $e(\xi)$ and $f(\xi)$ have the same meaning as in (3.7), and $\phi(\xi)$ is uniformly bounded in every interval $0 < \xi_0 \leq \xi \leq 1$. The solution $v(\xi)$ is given by the formula (3.9) with $r_{11}(t)$ replaced by $-\phi(t)t^{q_0}$. If

$$0 < m \leq |\pi(\xi)| \leq M \ , \quad \text{for } 0 \leq \xi \leq 1 \ , \tag{3.14}$$

we find

$$|v(\xi)| \leq \xi \ \frac{M}{m} \sup_{\xi \leq t \leq 1} |\phi(t)|(\xi^{-1} - 1) \leq c \sup_{\xi \leq t \leq 1} |\phi(t)|, \ 0 < \xi \leq 1,$$

provided $c \geq \frac{M}{m}$. If this result is combined with (3.13) we have, making also use of assumption (3.8), that

$$|v'(\xi)| \leq c\xi^{-1} \sup_{\xi \leq t \leq 1} |\phi(t)| \ , \quad 0 < \xi \leq 1 \ ,$$

if c is chosen sufficiently large.

We apply these inequalities to equation (2.13) which is of the form

$$\xi^{-q_o} e(\xi)x_s' + \xi^{-q_o-1} f(\xi)x_s = R_s^{(1)} \ ,$$

obtaining the inequalities

$$|x_s(\xi)| \leq c \ \xi^{q_o+1} \sup_{\xi \leq t \leq 1} |R_s^{(1)}| \tag{3.15}$$

$$|x_s'(\xi)| \leq c \ \xi^{q_o} \sup_{\xi \leq t \leq 1} |R_s^{(1)}| \tag{3.16}$$

4. The Convergence Proof for $q(0) > 0$.

Except for a few details our presentation so far follows that in [1] and [2] . In this section we supply the validity proof for the procedure, which is missing in those accounts.

We shall use the method of dominating series: First a dominating problem is constructed whose formal solution is given by series of positive terms that are term for term larger than the absolute values of the corresponding terms in the series solution of the original problem. Then the formal solution of the dominating problem is shown to converge, which implies the convergence of the original series.

We begin the description of the dominating problem by constructing a "dominating function" for the quantity h of (2.10). Let $k > 0$ be chosen so small that the functions $q(x)$, $r(x)$ are holomorphic in $|x| \leq 1 + k$. If K is an upper bound for $|q(x)|$ and $|r(x)|$ in that circle, then the series

$$\frac{K}{k^2}(x - \xi)^2 + \frac{K}{k^3}(x - \xi)^3 + \ldots = \frac{K}{k^2}(x-\xi)^2\left(1- \frac{x-\xi}{k}\right)^{-1}$$

dominates the series in powers of $x - \xi$ of the two expressions in brackets in formula (2.10). Hence the function

$$\widehat{h}(\widehat{x}, \widehat{u}, \widehat{x'}, \xi) = \frac{K}{k^2} (\widehat{x} - \xi)^2 (1 - \widehat{\frac{x-\xi}{k}})^{-1} (\widehat{u} + \widehat{x'}) \tag{4.1}$$

dominates h in the following sense: Expand (2.10) in powers of $x - \xi$, so that h becomes an infinite sum of terms of the form

$$A_{jmn}(\xi)(x - \xi)^j u^m x'^n; \quad j = 2, 3, \ldots; \quad m + n = 1 .$$

If

$$\widehat{A}_{jmn} (\widehat{x} - \xi)^j \widehat{u}^m \widehat{x'}^n$$

is the corresponding term in the analogous expansion of h, then

$$|A_{jmn} (\xi)| \leq A_{jmn} \qquad \text{in } |\xi| \leq 1 . \tag{4.2}$$

Now consider the formal series

$$\begin{aligned} \widehat{x} &= \xi + \widehat{x}_1 \alpha + \widehat{x}_2 \alpha^2 + \ldots \\ \widehat{x'} &= 1 + \widehat{x'_1} \alpha + \widehat{x'_2} \alpha^2 + \ldots \\ \widehat{u} &= \widehat{u}_o + \widehat{u}_1 \alpha + \widehat{u}_2 \alpha^2 + \ldots \\ \widehat{u'} &= \widehat{u'_o} + \widehat{u'_1} \alpha + \widehat{u'_2} \alpha^2 + \ldots \end{aligned} \tag{4.3}$$

where $\widehat{x'}$ and $\widehat{u'}$ are not the derivatives of $\widehat{x}$, $\widehat{u}$, but independent symbols whose numerical value is irrelevant at this

point. If these series are inserted into $\hat{h}$ and the series rearranged with respect to powers of α, then the coefficient $\hat{h}_s$ of α_s is a polynomial in the $\hat{x}_j$, $\hat{x}'_j$, $\hat{u}_j$ which dominates the polynomial h_s introduced before in analogous fashion.

Next we observe that the function

$$(\hat{x} - \xi)(\hat{u}' - \hat{u}'_o)$$

has the property that insertion of the series (4.3) produces a series in α whose s^{th} coefficient is $\sum_{\substack{\mu+\nu=s\\ \mu,\nu>0}} \hat{x}_\mu \hat{u}_\nu$, which is analogous to the first summation in (2.9). Similar arguments hold for the remaining terms of (2.9). Summarizing we get the following result: Let the constant K introduced above be chosen so large that also $|q'(x)| \leq K$ in $|\xi| \leq 1 + k$ and consider the function

$$\begin{aligned}\hat{R}(\hat{x},\hat{x}',\hat{u},\hat{u}';\xi;\alpha) &= (\hat{x}-\xi)(\hat{u}'-\hat{u}'_o) + \hat{\alpha}\hat{u}\hat{u}' + K(\hat{x}'-1)(\hat{u}-\hat{u}_o) \\ &+ K(\hat{x}-\xi)(\hat{x}'-1)\hat{u}' + \frac{K}{k}(\hat{x}-\xi)^2(k-(\hat{x}-\xi))^{-1}(\hat{u}+\hat{x}') \;.\end{aligned} \tag{4.4}$$

If $\hat{x}$, $\hat{x}'$, $\hat{u}$, $\hat{u}'$ are replaced by the series (4.3) then the coefficient $\hat{R}_s$ of α^s in the resulting series in powers of α is a polynomial in the $\hat{x}_j$, $\hat{x}'_j$, $\hat{u}_j$, $\hat{u}'_j$ with positive coef-

ficients, independent of ξ and not smaller than the absolute values of the corresponding coefficients of R_s, in $|\xi| \leq 1$, for $s = 1, 2, \ldots$.

In order to avoid an accumulation of complexities in the presentation, we give the remainder of the argument first for the special case that $R_s^{(2)} \equiv 0$ for <u>all</u> s . The modifications necessary if $R_s^{(2)} \not\equiv 0$, for $1 \leq s \leq m$, $m \geq 1$, will be indicated later. For our special case we have $u(\xi) = u_o(\xi)$. Let the constant c be chosen so large that in addition to the inequalities (3.15), (3.16) the relation

$$|u_o(\xi)| \leq c\xi^{-q_o}, \; |u_o'(\xi)| \leq c\xi^{-q_o-1} \; , \; 0 < \xi \leq 1 \qquad (4.5)$$

(cf. formula (3.4)) holds. If we define, furthermore, $\hat{u}_o$ by

$$\hat{u}_o = c\xi^{-q_o}$$

we can take as dominating problem

$$\hat{x} = \xi + c\xi^{q_o+1}\hat{R}(\hat{x}, \hat{x}\xi^{-1}, \hat{u}_o, \hat{u}_o\xi^{-1}; \xi, \alpha) \; . \qquad (4.6)$$

In fact, if $\hat{x}$ is replaced by $\xi + \alpha\hat{x}_1 + \ldots$ in (4.6) we find

$$\hat{x}_s = c\xi^{q_o+1}\hat{R}_s \quad ,$$

where in the polynomials $\hat{R}_s$ the quantities $\widehat{x'}_j$ must be replaced by $\hat{x}_j\xi^{-1}$, $\hat{u}_o$ by $c\xi^{-q_o}$, $\hat{u}'_o$ by $c\xi^{-q_o-1}$ and $\hat{u}_j$, $\widehat{u'}_j$ $(j > 0)$ by zero. In view of the dominating property of $\hat{R}_s$ and of the inequalities (3.15), (3.16) and (4.5) it follows inductively, that

$$|x_s(\xi)| \leq \hat{x}_s \text{ , } 0 < \xi \leq 1 \text{ .} \tag{4.7}$$

Thus the convergence of the series for $x(\xi)$ will be proved, if we show the convergence of the series $\xi + \hat{x}_1\alpha + \dots$, or, in other words, if we can show that equation (4.6) defines x as a holomorphic function of α near $\alpha = 0$. Written explicitly (4.6) reads

$$\hat{x} - \xi = c\xi^{q_o+1}[\alpha c^2\xi^{-2q_o-1} + K(\hat{x} - \xi)(\hat{x} - \xi)c\xi^{-q_o-1}$$
$$+ \frac{K}{k}(\hat{x} - \xi)^2(k-(\hat{x} - \xi))^{-1}(c\xi^{-q_o} + \hat{x}\xi^{-1})],$$

or, if we set $\hat{x} - \xi = \hat{z}$, $\alpha\xi^{-q_o} = \zeta$,

$$c^3\zeta + c^2K\hat{z}^2 + \frac{K}{k}\frac{\hat{z}^2c}{k-\hat{z}}(c\xi + \hat{z}\xi^{q_o} + \xi^{q_o+1}) - \hat{z} = 0 \text{ .} \tag{4.8}$$

The left side vanishes for $z = 0$, $\zeta = 0$. At this point its partial derivative with respect to $\hat{z}$ is different from

zero, in fact it is equal to -1 . Hence (4.8) can be solved for $\hat{z}$ leading to a holomorphic function of $\zeta = \alpha\xi^{-q_o}$ near $\zeta = 0$ with a radius of convergence that is bounded away from zero, for ξ in $0 \leq \xi \leq 1$. This proves that the series for $x(\xi)$ converges for

$$|\alpha| \leq \gamma\xi^{q_o} \quad ,$$

where γ is a positive constant independent of ξ . We summarize our results so far as a theorem.

Theorem 1. In the differential equation

$$(x + \alpha u)\frac{du}{dx} + q(x)u - r(x) = 0$$

the functions $q(x)$ and $r(x)$ are assumed to be holomorphic for $|x| \leq 1$. Let $u(x)$ be the solution for which $u(1) = b$. If

$$q_o = q(0) > 0 \quad ,$$

and if the solution $u_o(x)$ of the problem

$$x\frac{du_o}{dx} + q(x)u_o - r(x) = 0 \ , \quad u_o(1) = b \tag{4.9}$$

satisfies the condition

$$q(x)\, u_o(x) - r(x) \neq 0 \quad \text{in } 0 < x \leq 1$$

then

$$u(x) = u_o(\xi) , \tag{4.10}$$

where x and ξ are related by the equation

$$x = \xi + x_1(\xi)\alpha + x_2(\xi)\alpha^2 + \dots , \tag{4.11}$$

whose coefficients $x_s(\xi)$ can be calculated by formal insertion of (4.10) and (4.11) into (4.9) . The series (4.11) converges for

$$|\alpha| \leq \gamma\xi^{q_o} , \quad 0 \leq \xi \leq 1 \tag{4.12}$$

where γ is a certain positive constant.

Now we consider the more general case that

$$u(\xi) = u_o(\xi) + u_1(\xi)\alpha + \dots + u_m(\xi)\alpha^m \tag{4.13}$$

where the $u_j(\xi)$ have been determined by solving (2.14) with some admissible choice of $R_j^{(2)}$. This means that in our general procedure $R_s^{(1)} = R_s$ for $s > m$.

The constant c can be chosen so large that, in view of (3.11), (2.13) and (2.14)

$$|x_j| \leq c\,|\xi|^{-jq_o}, \quad |u_j| \leq c|\xi|^{-(j+1)q_o}, \quad |\xi| \leq 1, \; j \leq m, \qquad (4.14)$$

and also

$$|x_j'| \leq c|\xi|^{-jq_o-1}, |u_j'| \leq c|\xi|^{-(j+1)q_o-1}; |\xi| \leq 1, \; j \leq m. \qquad (4.15)$$

Hence, if we define

$$\hat{x}_j = c\xi^{-jq_o}, \quad \hat{u}_j = c\xi^{-(j+1)q_o}, \qquad j = 1,\ldots,m, \qquad (4.16)$$

we have

$$|x_j| \leq \hat{x}_j, \quad |u_j| \leq \hat{u}_j, \quad j = 1,\ldots,m; \quad 0 < \xi \leq 1. \qquad (4.17)$$

Let $\hat{u}$ be defined by

$$\hat{u} = \sum_{j=0}^{m} \hat{u}_j \alpha^j ;$$

then

$$\hat{u} = c\xi^{-q_o} \frac{1 - (\alpha\xi^{-q_o})^{m+1}}{1 - \alpha\xi^{-q_o}} . \qquad (4.18)$$

with the definitions $\widehat{x'} = \hat{x}\xi^{-1}$, $\widehat{u'} = \hat{u}\xi^{-1}$, i.e.,

$$\widehat{x'_s} = \hat{x}_s\xi^{-1}, \quad \widehat{u'_s} = \hat{u}_s\xi^{-1}, \qquad s = 1,2,\ldots, \tag{4.19}$$

it is furthermore true that

$$|x'_j| \leq \widehat{x'_j}, \quad |u'_j| \leq \widehat{u'_j}, \qquad j = 1,\ldots,m; \; 0 < \xi \leq 1. \tag{4.20}$$

Setting, as before,

$$\hat{z} = \hat{x} - \xi, \qquad \zeta = a\xi^{-q_o},$$

the function $\hat{R}$ of (4.4) becomes now, after some simplifications,

$$\hat{R} = \xi^{-q_o-1}\left\{c(1+K)\hat{z}\zeta\,\frac{1-\zeta^m}{1-\zeta} + c^2\zeta\left(\frac{1-\zeta^{m+1}}{1-\zeta}\right)^2\right.$$

$$\left. + cK\hat{z}^2\,\frac{1-\zeta^{m+1}}{1-\zeta} + \frac{K}{k}\,\frac{\hat{z}^2}{k-\hat{z}}\left(c\xi\,\frac{1-\zeta^{m+1}}{1-\zeta} + \hat{z}\xi^{q_o} + \xi^{q_o+1}\right)\right\}. \tag{4.21}$$

However, this is not a correct choice of $\hat{R}$ to use in (4.6) in the present case, since the values obtained for $\hat{x}_1,\ldots,\hat{x}_m$ in this manner would be different from those defined in (4.16) and might fail to satisfy the first inequa-

lity in (4.17). In order to construct a dominating function with the desired properties assume first that $m = 1$ and let $\phi_1(\xi)$ be the function obtained for $\hat{x}_1$ instead of $c\xi^{-q_o}$ from the equation

$$\hat{z} = c\xi^{q_o + 1}\hat{R}, \tag{4.22}$$

$\hat{R}$ being defined by (4.21). Then the same calculation repeated with

$$\hat{z} = c\xi^{q_o + 1}\hat{R} + \alpha(c\xi^{-q_o} - \phi_1(\xi)) \tag{4.23}$$

does yield for $\hat{x}_1$ the required value $c\xi^{-q_o}$. The function $\phi_1(\xi)$ is of the form

$$\phi_1(\xi) = \xi^{-q_o}\phi_1^*(\xi) \tag{4.24}$$

with $\phi_1^*(\xi)$ bounded in $0 \leq \xi \leq 1$. This can be shown by the same argument as that leading to Theorem 1, with $\hat{R}$ now defined by (4.21). Hence, equation (4.23) can be written

$$\hat{z} = c\xi^{q_o + 1}\hat{R} + (c - \phi_1^*(\xi))\zeta. \tag{4.25}$$

The value $\phi_2(\xi)$ obtained instead of $c\xi^{-2q_o}$ for $\hat{x}_2$ from this equation is of the form $\phi_2(\xi) = \xi^{-2q_o}\phi_2^*(\xi)$ with bounded $\phi_2^*(\xi)$. In order to obtain the correct value for $\hat{x}_2$, formula (4.25) must therefore be replaced by

$$\hat{z} = c\xi^{q_o+1}\hat{R} + (c - \phi_1^*(\xi))\zeta + (c - \phi_2^*(\xi))\zeta^2. \qquad (4.26)$$

This change does not affect the value of $\hat{x}_1$. Continuing in this fashion an equation

$$\hat{z} = c\xi^{q_o+1}\hat{R} + \sum_{j=1}^{m} (c - \phi_j^*(\xi))\zeta^j$$

is finally obtained that is a dominating equation for our problem. This proves

<u>Theorem 2</u>. <u>Theorem 1 can be generalized as follows: The solution</u> $u(x)$ <u>can be represented parametrically in the form</u>

$$u(\xi) = u_o(\xi) + u_1(\xi)\alpha + \dots + u_m(\xi)\alpha^m$$

$$x(\xi) = \xi + x_1(\xi)\alpha + x_2(\xi)\alpha^2 + \dots,$$

<u>where</u> m <u>is an arbitrary non-negative integer. The coeffi-</u>

cients $u_s(\xi)$, $x_s(\xi)$ are to be calculated according to the formal procedure explained in sections 2 and 3, setting $R_s^{(2)} = 0$ for $s > m$.

The series for $x(\xi)$ converges for

$$|\alpha| \leq \gamma_m \xi^{q_o}, \quad 0 \leq \xi \leq 1$$

γ_m being a certain constant.

Corollary: A slightly weaker variant of the statement of Theorem 2 consists in saying that any admissable way of splitting R_s into $R_s^{(1)}$ and $R_s^{(2)}$ in applying the method of sections 2 and 3 leads to series that are asymptotic to the solution $x(\xi)$, $u(\xi)$ of the differential equation in the sense that

$$x(\xi) = \xi + \sum_{s=1}^{m} x_s(\xi)\alpha^s + \alpha^{m+1}\xi^{-(m+1)q_o} E_m(\xi,\alpha)$$

$$u(\xi) = \sum_{s=o}^{m} u_s(\xi)\alpha^s + \alpha^{m+1}\xi^{-(m+2)q_o} F_m(\xi,\alpha)$$

with $E_m(\xi,\alpha)$ and $F_m(\xi,\alpha)$ uniformly bounded for

$$|\alpha\xi^{q_o}| \leq \gamma_m, \quad 0 \leq \xi \leq 1 .$$

5. Supplementary Remarks

a) The point $x = 0$. The main purpose of Lighthill's construction is to find an expansion that can be used in the neighborhood of $x = 0$, $\alpha = 0$. In order to show that this is the case we return to the form in which our main result was first obtained. We had found a function

$$z = F(\xi,\zeta) \quad , \tag{5.1}$$

where $z = x - \xi$, holomorphic in ζ for $|\zeta| \leq \gamma_m$, $0 \leq \xi \leq 1$. The number γ_m is independent of ξ. The dependence on ξ is holomorphic in $0 < \xi \leq 1$ and continuous in $0 \leq \xi \leq 1$. Equation (5.1) is satisfied for $z = \zeta = 0$. Hence, the inverse function

$$\zeta = G(\xi,z) \tag{5.2}$$

exists and is a holomorphic function of z in a neighborhood of $z = 0$, for values of ξ in $0 \leq \xi \leq 1$ such that $[\partial F(\xi,\zeta)/\partial\zeta]_{\zeta=0} = x_1^*(\xi) \neq 0$. If complications are to be avoided we must introduce the hypothesis

$$x_1^*(0) \neq 0 \ . \tag{5.3}$$

(The case $x_1^*(0) = 0$ is briefly discussed in [1], p. 1186).
Then $x_1^*(\xi) \neq 0$ for $0 \leq \xi \leq \xi_1 \leq 1$. Since the branchpoints ζ where $\partial F/\partial\zeta = 0$ depend continuously on ξ, there is a number γ_m^* such that the circle $|\zeta| \leq \gamma_m^*$ contains no such branchpoint for $0 \leq \xi \leq \xi_1$. The image of this circle depends continuously on ξ and contains the point $z = 0$ in its interior. It follows that, although the circle of regularity of the function $G(\xi,z)$ about $z = 0$ may depend on ξ, it has a positive lower bound δ_m, for $0 \leq \xi \leq \xi_1$. The series for $G(\xi,z)$ converges, in particular, for $z = -\xi$, provided

$$0 < \xi \leq \min(\xi_1,\delta_m). \tag{5.4}$$

If ζ_o is the value of ζ in (5.2) corresponding to $z = -\xi$ we get, by inversion,

$$-\xi = F(\xi,\zeta_o) = \sum_{s=1}^{\infty} x_s^*(\xi)\zeta_o^s, \tag{5.5}$$

i.e., $x = 0$. In other words, x vanishes for all combinations of values of ξ and α such that

$$\alpha\xi^{-q_o} = \zeta_o$$

and that (5.4) is satisfied.

Since $G(\xi,z) = \dfrac{1}{x_1^*(\xi)} z + \sum_{s=2}^{\infty} g_s(\xi)z^s$ ($g_s(\xi)$ holomorphic in ξ and ξ^{q_o}),

we have

$$\zeta_o = G(\xi, -\xi) = -\xi/x_1^*(\xi) + \sum_{s=2}^{\infty} g_s(\xi)(-\xi)^s$$

and therefore, if α and ξ are related by $\alpha\xi^{-q_o} = \zeta_o$,

$$\alpha = -\xi^{q_o+1}/x_1^*(\xi) + \xi^{q_o} \sum_{s=2}^{\infty} g_2(\xi)(-\xi)^s .$$

The right member is a once continuously differentiable function of the variable $t = \xi^{q_o+1}$ near $t = 0$. By an application of the implicit function theorem we conclude that

$$\xi^{q_o+1} = - x_1^*(o)\alpha + o(\alpha)$$

i.e.

$$\xi = (- x_1^*(o)\alpha)^{1/q_o+1)} + o(\alpha). \tag{5.6}$$

This value of ξ can be taken real if $x_1^*(0) < 0$, and it follows that our series for $x(\xi)$ does indeed assume the value zero for a certain positive ξ, if α is small enough. If $x_1^*(0) > 0$, the solution $x(\xi)$ of our differential equation problem does not assume the value zero in $0 \leq \xi \leq 1$. Further discussion of this situation can be found in [1].

b) <u>The initial condition.</u> In an example given in [1] and [2] our rigid choice of the boundary conditions as

$$x_s(1) = 0, \quad u_o(1) = b; \quad u_s(1) = 0 , \qquad s = 1,2,\ldots,$$

is replaced by the more general one:

$$\xi_1 + \sum_{s=1}^{\infty} x_s(\xi_1)\alpha^s = 1, \qquad \sum_{s=0}^{\infty} u_s(\xi_1)\alpha^s = b,$$

which leads to a recursive set of relations between the $x_s(\xi_1)$ and $u_s(\xi_1)$. This method has the advantage that half of the initial values can be chosen arbitrarily in the course of the calculation so as to simplify the resulting expression. An extension of the argument in section 4 which would include this possibility can very likely be given, but it will not be attempted here.

c) <u>A limitation of the method.</u> It is stated in [1] that the method extends without essential modifications to differential equations of the form

$$[x + \alpha p_1(x,u) + \ldots] \frac{du}{dx} + q(x)u = r(x) + \alpha r_1(x,u) + \ldots .$$

I fail to see how this can be maintained in such generality. If, for instance

$$r_1(x,u) = u^3$$

one finds

$$R_1 = u_o u_o' + u_o^3 = O(\xi^{-3q_o}) ,$$

instead of $O(\xi^{-2q_o-1})$. Thus a non-trivial modfication of the procedure seems to be necessary. Lighthill's claim is correct if $p_s(x,u)$ and $r_s(x,u)$ are polynomials of degree s at most in u, but this would exclude an equation as simple as

$$(x + \alpha u)\frac{du}{dx} + u = \alpha \sin u.$$

6. <u>The Convergence Proof for</u> $q_o < 0$. Leaving aside the case $q_o = 0$ in which logarithmic terms occur in the series expansion but which offers no essential new difficulties (cf. [1] and [2] we assume now that

$$q_o = -\kappa < 0 .$$

The convergence of Lighthill's procedure in this case could very likely be proved by the methods of this article. Instead we shall treat this problem by a variant of Lighthill's technique that avoids some of the difficulties he encounters.

Every solution of the system

$$\xi x' = x + \alpha y$$
$$\xi u' = r(x) - q(x)u \tag{6.1}$$

satisfies the single equation (1.1). The converse is not true. By solving, as we small, this system instead of (1.1) we lose, therefore, some of the flexibility of Lighthill's method. This flexibility could be regained by multiplying the right members of (6.1) by some function

$$F(t,\alpha) = 1 + F_1(t)\alpha + F_2(t)\alpha^2 + \ldots$$

and by determining the $F_s(t)$ successively in the course of the calculation. (Compare the appendix by Erdélyi in [2] in this regard). But we shall treat only equation (6.1) as given.

As pointed out by Lighthill, there is some simplification of the argument, if the property

$$r(0) = 0 \tag{6.2}$$

is secured by means of a preliminary translation of the variables. This will be assumed to have been done.

If K is not an integer, the solution $u_0(\xi)$ of the differential equation problem

$$\xi u_0' = r(\xi) - q(\xi)u_0, \qquad u_0(1) = b$$

is then of the form

$$u_o(\xi) = \xi^{\kappa}p_1(\xi) + \xi p_2(\xi) = \xi^{\rho}p^*(\xi) \tag{6.3}$$

where $p_1(\xi)$, $p_2(\xi)$ are holomorphic for $0 \leq \xi \leq 1$,

$$\rho = \min(\kappa, 1), \tag{6.4}$$

and $p^*(\xi)$ holomorphic in $0 < \xi \leq 1$, continuous in $0 \leq \xi \leq 1$. If κ is an integer a term of the form ξ log ξ appears. The proof is not seriously complicated by this term. In the interest of simplicity we shall, however, assume that κ is not an integer.

Setting

$$u = u_o + y \tag{6.5}$$

the system (6.1) is converted into

$$\xi x' = x + \alpha u_o(\xi) + \alpha y \tag{6.6}$$

$$\xi y' = -q(\xi)y + R(x,y,\xi) \tag{6.7}$$

with

$$R(x,y,\xi) = r(x) - r(\xi) - [q(x) - q(\xi)][u_o + y]. \tag{6.8}$$

We insert

$$x = \xi + \sum_{s=1}^{\infty} x_s\alpha^s, \quad y = \sum_{s=1}^{\infty} u_s\alpha^s \tag{6.9}$$

into (6.6) and (6.7). This leads, in the usual manner, to the recursion formulas

$$\xi x_s' = x_s + u_{s-1}, \qquad x_s(1) = 0, \qquad s > 0 \tag{6.10}$$

$$\xi u_s' = -qu_s + h_s, \qquad u_s(1) = 0, \qquad s > 0 \tag{6.11}$$

where

$$h_s = h_s(x_1, \ldots, x_s; u_1, \ldots, u_{s-1}; \xi) . \tag{6.12}$$

The solutions of (6.10) and (6.11) are

$$x_s(\xi) = \xi \int_1^{\xi} \tau^{-2} u_{s-1}(\tau)\, d\tau , \tag{6.13}$$

$$u_s(\xi) = \xi^{\kappa} p(\xi) \int_1^{\xi} \tau^{-\kappa-1} p^{-1}(\tau) h_s(\tau)\, d\tau . \tag{6.14}$$

Here $p(\xi)$ is holomorphic and different from zero for $0 \leq \xi \leq 1$ and $h_s(\tau)$ is an abbreviated notation for the value of h_s in (6.12) at $\xi = \tau$. Hence, we have the appraisals

$$|x_s(\xi)| \leq \max_{\xi \leq \tau \leq 1} |u_{s-1}(\tau)| , \qquad s > 0,\ 0 \leq \xi \leq 1 \tag{6.15}$$

$$|u_s(\xi)| \leq c\xi^{\kappa} \max_{\xi \leq \tau \leq 1} |h_s(\tau)|, \qquad s > 0, \quad 0 \leq \xi \leq 1 \tag{6.16}$$

with certain constant c, which we choose so large that, in addition, (6.3) implies

$$|u_o(\xi)| \leq c\xi^\rho \; . \tag{6.17}$$

The last three inequalities form the basis of a convergence proof for the series (6.9). The function

$$\hat{R}(x,y,\xi) = K \frac{x - \xi}{k-(x - \xi)} (1 + c\xi^\rho + y) \tag{6.18}$$

is readily seen to dominate the function $R(x,y,\xi)$ of (6.8). Therefore the equations

$$\hat{x} = \xi + a(c\xi^\rho + \hat{y}) \tag{6.19}$$

$$\hat{u} = c\xi^K \hat{R}(\hat{x},\hat{y},\xi) \tag{6.20}$$

constitute a problem that dominates the differential equation problem (6.6), (6.7) with $x(1) = 1$, $y(1) = 0$ in the previously explained sense, Insertion of (6.19) into (6.20) and use of the definition (6.18) yield

$$\hat{u} = c\xi^K \frac{a(c\xi^\rho+\hat{u})}{k - a(c\xi^\rho+\hat{u})} (1 + c\xi^\rho + \hat{u}). \tag{6.21}$$

This equation defines $\hat{u}$ implicitly as a holomorphic function of the variable

$$\eta = a\xi^\rho \tag{6.22}$$

in a circle

$$|\eta| \leq \delta$$

where δ can be taken independent of ξ. The relation (6.19) shows then that $\hat{x} - \xi$, too, is holomorphic in η. Thus the following theorem has been proved.

Theorem 3. If $q(0)$ is negative and $r(0) = 0$ the solution of $(x + \alpha u)\frac{du}{dx} + q(x)u - r(x) = 0$ with $u(1) = b$ can be represented by two series of the form

$$x = \xi + \sum_{s=1}^{\infty} x_s(\xi)\alpha^s, \quad u = u_o(\xi) + \sum_{s=1}^{\infty} u_s(\xi)\alpha^s$$

whose coefficients are calculated by formal insertion into (6.1). The series converge for

$$|\alpha| \leq \delta\xi^{-\rho}, \quad \rho = \min(\varkappa, 1), \quad 0 \leq \xi \leq 1,$$

where δ is a certain positive constant.

Thanks to assumption (6.2) we have $x_s(0) = 0$. Hence $x(0) = 0$, and the preceding theorem gives us a parametric representation of the solution curve valid between $x = 1$, $u = b$ and $x = 0$, $u = 0$.

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